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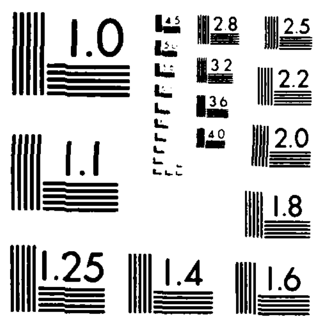

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Variance-stable R-estimators

by

Elvezio Ronchetti and James H. Yen

Technical Report No. 266, Series 2

Department of Statistics

Princeton University

May 1984

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Variance-stable R-Estimators

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SUMMARY

By means of the concept of change-of-variance function we investigate the stability properties of the asymptotic variance of R-estimators. This allows us to construct the optimal V-robust R-estimator that minimizes the asymptotic variance at the model, under the side condition of a bounded change-of-variance function. Finally, we discuss the connection between this function and an influence function for two-sample rank tests introduced by Eplett (1980).

Keywords: R-estimators, influence function, change-of-variance function, robust estimation, asymptotic variance.

The first author was supported by ARO (Durham) contract # DAAG29-82-K-0178.

1. INTRODUCTION

R-estimators, that is estimators derived from rank tests, have become popular mostly because they require fewer distributional assumptions than do more classical procedures. In particular, R-estimators inherit some robustness properties from the rank tests from which they are derived. However, the degree of robustness varies among those estimators and depends on the properties one is interested in.

In this paper we are primarily concerned with the stability aspects of the (asymptotic) variance of R-estimators. Clearly this is an important aspect from a robustness point of view because it is strongly connected with the stability of confidence intervals.

In Section 2 we review the concepts of influence function and change-of-variance function. The latter is used in Section 3 to investigate the robustness properties of the asymptotic variance of R-estimators, and to approximate the asymptotic variance over neighborhoods of the assumed central model. Moreover, a connection between the change-of-variance function and an influence function for two-sample rank tests due to Eplett (1980) is established. Finally in Section 4 we find the optimal V-robust R-estimator, that is, an R-estimator that satisfies the first order necessary conditions for being a minimum of the asymptotic variance at the model, subject to a bound on the change-of-variance function.

2. THE INFLUENCE FUNCTION AND THE CHANGE-OF-VARIANCE FUNCTION

In this section we review briefly two basic tools that have been used successfully to study the robustness properties of estimators.

The first one, the influence function, was introduced by Hampel (1968, 1974) and is essentially the first derivative of an estimator viewed as functional. It describes the normalized influence of an infinitesimal observation on the estimator. Formally, suppose the estimator T_n can be expressed as a function T of the empirical distribution function F_n, $T_n = T(F_n)$. Then the *influence function* of T at F is given by

$$IF(x;T,F) = \lim_{\epsilon \to 0} [T((1-\epsilon)F + \epsilon\delta_x) - T(F)] / \epsilon , \tag{2.1}$$

where δ_x is the distribution that puts mass 1 at the point x. From a robustness point of view, a desirable property of this function is boundedness.

Under suitable regularity conditions, we can approximate T(G) for G close to F (in some metric) by

$$\begin{aligned} T(G) &\cong T(F) + \int IF(x;T,F)\, d(G-F)(x) \\ &= T(F) + \int IF(x;T,F)\, dG(x). \end{aligned} \tag{2.2}$$

The two terms on the right hand side of (2.2) are the first two components of the von Mises expansion of T; cf. von Mises (1947). Moreover, the asymptotic variance of T can be computed as (Hampel 1968, 1974)

$$V(T,F) = \int IF^2(x;T,F)\, dF(x). \tag{2.3}$$

From (2.2) we can also derive an approximation for the maximum asymptotic bias over a "neighborhood" $P_\epsilon(F) = \{G \mid F = (1-\epsilon)F + \epsilon H$, H arbitrary $\}$ caused by the contamination, namely

$$sup\{ \mid T(G) - T(F) \mid : G \in P_\epsilon(F) \} \cong \epsilon \cdot \gamma^*(T,F), \tag{2.4}$$

where $\gamma^*(T,F) = \sup_x \mid IF(x;T,F) \mid$ is the so-called *gross-error sensitivity*.

The second tool is the *change-of-variance function*. It can be viewed as the derivative of the asymptotic variance V(T,F), and therefore describes its infinitesimal stability. Formally,

$$CVF(x;T,F) = \lim_{\epsilon \to 0} [V(T,(1-\epsilon)F + \epsilon\delta_x) - V(T,F)] / \epsilon \tag{2.5}$$

and

$$\kappa^*(T,F) = \sup_x CVF(x;T,F) / V(T,F); \tag{2.6}$$

cf. Hampel (1973); Hampel, Rousseeuw, Ronchetti (1981); Rousseeuw (1981) and, without the standardization on the right hand side of (2.6), Ronchetti (1979). κ^* is called *change-of-variance sensitivity*. Note that dividing the CVF by V(T,F) in (2.6) amounts to measuring the influence on the asymptotic variance in the logarithmic scale which is natural when working with variances; cf. Wegman and Carroll (1977) and Hampel, Rousseeuw, Ronchetti (1981).

Using an expansion similar to (2.2) we can obtain an approximation to the maximum variance over $P_\epsilon(F)$, namely (Hampel 1983)

$$sup\{V(T,G) : G \in P_\epsilon(F)\} \cong V(T,F) \cdot \exp\{\epsilon \cdot \kappa^*(T,F)\}. \tag{2.7}$$

We call an estimator T *B-robust* iff $\gamma^*(T,F) < \infty$ and *V-robust* iff $\kappa^*(T,F) < \infty$. Special attention will be given to V-robustness in later sections of this paper.

The change-of-variance function has a strong connection with an influence curve for two-sample rank tests introduced by Eplett (1980). We investigate this relationship in Section 3.

Existence conditions and mathematical properties of derivatives of functionals including the influence function are discussed extensively in Fernholz (1983). For a complete account of the approach to robust statistics using influence functions, we refer to Hampel, Ronchetti, Rousseeuw, Stahel (1984).

3. ROBUSTNESS PROPERTIES OF R-ESTIMATORS

Let $x_1, \ldots, x_n$ be n independent identically distributed observations and consider the location model $F(\cdot-\theta)$. We shall be concerned with the robustness properties of R-estimators T_n of θ which are derived from two-sample rank tests with some score generating function J. To compute T_n, we first construct a mirror image of $2T_n - x_1, \ldots, 2T_n - x_n$ of the original sample and then find T_n such that the rank test is least able to detect a difference in location; cf. Huber (1981).

We assume the following conditions on the model distribution F and on the score generating function J:

(F) F has an absolutely continuous symmetric density f. Denote by f' its derivative.

(J) $J : [0,1] \to \mathbb{R}$ is continuous, non-decreasing and square integrable. Moreover, J(1-u) = -J(u) for all u $\in$ [0,1].

(JF) $\psi_F = J \circ F$ is differentiable on $\mathbb{R} \setminus D(\psi_F, F)$, where D is a finite set, and the derivative ψ_F' is bounded.

Conditions (F) and (J) are standard conditions when dealing with R-estimators; cf. for instance, Hájek and Šidák (1967). In addition we require (JF) to ensure the existence of the change-of-variance function (see below); cf. also Jaeckel (1971).

Under the given conditions, the R-estimator T_n derived from a two-sample rank test with score generating function J, converges in probability to T(F), where T(F) is the solution of the equation (Huber 1981)

$$\int J([F(x)+1-F(2T(F)-x)]/2)\, dF(x) = 0. \tag{3.1}$$

From (3.1) by replacing F with $(1-\epsilon)F + \epsilon\delta_x$, and taking the derivative with respect to ϵ at $\epsilon = 0$, one can easily compute the influence function of the R-estimator T defined by J

$$IF(x;J,F) = \psi_F(x)/B(\psi_F,F), \tag{3.2}$$

where $\psi_F(x) = J(F(x))$ and $B(\psi_F,F) = \int \psi_F'(x)dF(x)$. Moreover, from (2.3) with $A(\psi_F,F) = \int \psi_F^2(x)\, dF(x)$, we obtain

$$V(J,F) = A(\psi_F,F)/B^2(\psi_F,F), \tag{3.3}$$

and from (2.5)

$$CVF(x;J,F) = 2V(J,F)\cdot[1+(h_F(x;\psi_F)-\psi_F'(x))/B(\psi_F,F)] \tag{3.4}$$

for all $x \in \mathbb{R} \setminus D(\psi_F,F)$, where

$$h_F(x;\psi_F) = \left(\int_{-\infty}^{-x} + \int_{-\infty}^{x}\right) \psi_F'(u)[-f'(u)/f(u)]\cdot F(u)du \;; \tag{3.5}$$

cf. Ronchetti (1979), Hampel (1983).

Remarks

1) Eplett (1980) defines an influence curve for two-sample rank tests with score generating function J by taking their asymptotic power β(J,F) and computing an influence function on β. Since $\beta(J,F) = (V(J,F))^{-1/2}$, where V is the asymptotic variance of the corresponding R-estimators, the change-of-variance function given by (3.4) is proportional to Eplett's influence curve.

2) Denote by $T^{(R)}$ the R-estimator defined by J, and by $T^{(M)}$ the M-estimator of location defined by ψ_F, i.e. $T_n^{(M)}$ is the solution of the implicit equation

$$\sum_{i=1}^{n} \psi_F (x_i - T_n^{(M)}) = 0.$$

From (3.2) it follows that $T^{(R)}$ and $T^{(M)}$ have the same influence function and therefore the same asymptotic variance at F. However, the change-of-variance functions are different and the following holds:

$$CVF(x; T^{(R)}, F) = CVF(x; T^{(M)}, F) +$$
$$V(J, F)[1+2h_F(x; \psi_F)/ B(\psi_F, F) - \psi_F^2(x)/ A(\psi_F, F)].$$

It turns out (see examples below) that in general the change-of-variance sensitivity of R-estimators is smaller than that of the corresponding M-estimators.

Let us now investigate the V-robustness of some R-estimators at the normal model. We denote by Φ and ϕ the distribution function and the density of the standard normal distribution, respectively.

The *Hodges-Lehmann estimator* is defined by $J_{HL}(u) = u-½$. Therefore, $CVF(x; J_{HL}, \Phi)/ V(J_{HL}, \Phi) = 4-8\phi(x)\sqrt{\pi}$ and $\kappa^*(J_{HL}, \Phi) = 4$. So the Hodges-Lehmann estimator is V-robust.

The *normal scores estimator* is defined by $J_{NS}(u) = \Phi^{-1}(u)$ and is the most efficient R-estimator at the normal model (F=Φ); cf. Hájek and Šidák (1967).
However, $CVF(x; J_{NS}; \Phi) = x^2-1$ and $\kappa^*(J_{NS}, \Phi) = \infty$. This shows the lack of stability in the asymptotic variance of this estimator.

The *bounded normal scores estimator* is defined by

$$J_c(u) = \begin{cases} \Phi^{-1}(u) & \text{if } |\Phi^{-1}(u)| \leqslant c \\ c \cdot \mathit{sign}\ \Phi^{-1}(u) & \mathit{otherwise} \end{cases} \qquad (3.6)$$

for some positive constant c. It is the most efficient R-estimator at the normal model *with a bounded influence function;* cf. Rousseeuw and Ronchetti (1979), Hampel (1983). Its change-of-variance function is a truncated (from above) parabola and its change-of-variance sensitivity is given by

$$\kappa^*(J_c, \Phi) = 1+c^2+2c\,\phi(c)/[2\Phi(c)-1]. \qquad (3.7)$$

4. MOST V-ROBUST AND OPTIMAL V-ROBUST R-ESTIMATORS

From Section 3 we know that the normal scores estimator is the most efficient R-estimator at the normal model but, unfortunately, has an unbounded change-of-variance function. This section will be devoted to finding an R-estimator that compromises in an optimal way efficiency at the model and V-robustness. We shall show that the bounded normal scores estimator (see Section 3) satisfies the first order necessary conditions for being a minimum of the asymptotic variance at the normal model, subject to the condition of a bounded change-of-variance function.

We prove the results of the section for the normal model ($F = \Phi$), but everything goes through for *model* distributions F which are "similar" to the normal, i.e. for F that satisfy (F) and $d^2(-\log f(x))/dx^2 > 0$ for all $x \in \mathbb{R}$. To simplify notation we drop the subscript F and write ψ for ψ_Φ , $B(\psi)$ for $B(\psi_\Phi, \Phi)$, etc.

The first theorem establishes a lower bound for the change-of-variance sensitivity of an R-estimator.

Theorem 1

Assume conditions (J) and (JF). Then $\kappa^*(J, \Phi) \geq 2$, and the median reaches this lower bound and is therefore the most V-robust R-estimator.

Proof. By (3.4) we have to prove that

$$sup\,\{h(x;\psi) - \psi'(x) : x \in \mathbb{R} \setminus D(\psi)\} \geq 0\,, \tag{4.1}$$

where

$$h(x;\psi) = \left(\int_{-\infty}^{-x} + \int_{-\infty}^{x}\right) \psi'(u) u\, \Phi(u) du.$$

Because of (J) and (JF), ψ is odd and monotone non-decreasing ($\psi'(x) \geq 0$). Therefore, h is symmetric and monotone non-decreasing on $[0,\infty)$ ($h'(x;\psi) = \psi'(x) \cdot x \geq 0$ for all $x \geq 0$). Since by (3.4) $\int h(x;\psi) d\Phi(x) = 0$, there exist $x_0 > 0$ such that $h(x) \leq 0$ *on* $[0, x_0]$ and $h(x) > 0$ on (x_0, ∞). Let us consider two cases.

Case 1. Suppose there exists $x_1 > x_0$ such that $\psi'(x_1) = 0$. Then, $h(x_1;\psi) - \psi'(x_1) = h(x_1) > 0$ and therefore (4.1) holds.

Case 2. Suppose that $\psi'(x) > 0$ for all $x > x_0$. This implies that $h(x;\psi)$ is strictly positive and monotone increasing on (x_0, ∞). Since ψ' is bounded, by (JF), there exists M such that $\psi'(x) \leq M$ for all x.

Case 2.1. $h(x;\psi)$ tends to ∞ as $x \to \infty$. Then there exists $x_2 > x_0$ such that $h(x_2;\psi) > M \geq \psi'(x_2)$, and (4.1) holds.

Case 2.2. $h(x;\psi)$ does not tend to ∞ as $x \to \infty$. Since $h'(x;\psi) > 0$, $h(x;\psi)$ must approach a bound asymptotically. Therefore, $h'(x;\psi) \to 0$ as $x \to \infty$ and, since $h(x;\psi)$ is strictly positive on (x_0, ∞), (4.1) follows.

From Rousseeuw (1981) we know that the change-of-variance sensitivity of the median (which is as well an R-estimator) equals 2 at the normal model. Therefore, the median reaches the lower bound in the class of R-estimators as well. This completes the proof.

Theorem 1 states that the median is the best R-estimator in terms of V-robustness. Unfortunately, its efficiency at the normal model is quite low ($= 2/\pi \simeq 64\%$). Therefore, a better compromise between V-robustness and efficiency is needed.

The following Lemma will be used in the proof of the optimality result given in Theorem 3.

Lemma

Let $\eta(c) = \kappa^*(J_c, \Phi)$ be the change-of-variance sensitivity of the bounded normal scores estimator. Then η is a bijection from $(0,\infty)$ onto $(2,\infty)$.

Proof. From (3.7) we have: $\eta(c) = 1+c^2+2c\,\phi(c)/[2\Phi(c)-1]$. Taking the derivative, we obtain

$$\eta'(c) = \alpha(c)/[2\Phi(c)-1]^2,$$

where $\alpha(c) = 2c\,[2\Phi(c)-1]^2+2\phi(c)[(1-c^2)(2\Phi(c)-1)-2c\,\phi(c)]$. An easy calculation shows that $\alpha'(c) > 0$ for all $c > 0$), and $\alpha(0) = \alpha'(0) = 0$. Therefore, $\alpha(c) > 0$ for all $c > 0$, and this implies that η is monotone increasing on $(0,\infty)$. Moreover, from the definition of η, it is clear that $\eta(c) \to 2$ as $c \to 0$, and $\eta(c) \to \infty$ as $c \to \infty$. This completes the proof.

Our optimality result is based on a theorem that can be found in Hestenes (1966), p. 265. For the sake of completeness, we rewrite it in our notation. (Note the obvious misprint in formula (5.7), p. 263)

Theorem 2

Let C be the class of piecewise continuous differentiable vector functions with l components and defined on some interval (a,b). Denote by ξ' the derivative of any $\xi \in C$. Suppose $\xi^{(0)}(x) = (\xi_1^{(0)}(x), \ldots, \xi_l^{(0)}(x))$ minimizes

$$I(\xi) = \int_a^b L(x, \xi(x), \xi'(x))\,dx \tag{4.2}$$

among all ξ in C satisfying the constraints

$$\gamma_k(x, \xi, \xi') \leq 0, \qquad 1 \leq k \leq m', \tag{4.3}$$

$$\gamma_k(x, \xi, \xi') = 0, \qquad m' < k \leq m, \tag{4.4}$$

the fixed endpoint conditions

$$\xi_i(a) = X_a^{(i)}, \quad \xi_i(b) = X_b^{(i)}, \qquad 1 \leq i \leq l, \tag{4.5}$$

and the isoperimetric conditions

$$I_j(\xi) = \int_a^b L_j(x, \xi(x), \xi'(x))\,dx = 0, \quad 1 \leq j \leq p. \tag{4.6}$$

Then there exist multipliers

$$\lambda_0 \geq 0,\ \lambda_j,\ \mu_k(x), \qquad 1 \leq j \leq p,\ 1 \leq k \leq m,\ a \leq x \leq b$$

not vanishing simultaneously on (a,b) and a function

$$G(x, \xi, \xi', \mu) = \lambda_0 L + \sum_{j=1}^{p} \lambda_j L_j + \sum_{k=1}^{m} \mu_k \gamma_k \tag{4.7}$$

such that

(i) The multipliers $\mu_k(x)$ are continuous between corners of $\xi^{(0)}$;

(ii) The functions $G_{\xi_i'}$, $G - \sum_{i=1}^{l} \xi_i' G_{\xi_i'}$, with $\mu_k = \mu_k(x)$, are continuous along $\xi^{(0)}$ and satisfy the relations

$$\frac{d}{dx} G_{\xi_i'} = G_{\xi_i}, \quad 1 \leqslant i \leqslant l, \quad \frac{d}{dx}(G - \sum_{i=1}^{l} \xi_i' G_{\xi_i'}) = G_x \tag{4.8}$$

between corners of $\xi^{(0)}$;

(iii) The inequality

$$E(x, \xi^{(0)}(x), \xi^{(0)\prime}(x), \mu, \mu(x)) \geqslant \sum_{k=1}^{m} \mu_k(x) \cdot \gamma_k(x, \xi^{(0)}, \mu) \tag{4.9}$$

holds for all u such that

$$\gamma_k(x, \xi^{(0)}(x), \mu) \leqslant 0, \quad 1 \leqslant k \leqslant m', \tag{4.10}$$

$$\gamma_k(x, \xi^{(0)}(x), \mu) = 0, \quad m' \leqslant k \leqslant m, \tag{4.11}$$

where E is the Weierstrass E-function

$$E(x, \xi, \xi', \mu, \mu) = G(x, \xi, \mu, \mu) - G(x, \xi, \xi', \mu) - \sum_{i=1}^{l} (u_i - \xi_i') G_{\xi_i'}(x, \xi, \xi', \mu). \tag{4.12}$$

In addition, $\mu_k(x) \geqslant 0$ $(1 \leqslant k \leqslant m')$ with $\mu_k(x) = 0$ whenever $\gamma_k(x, \xi^{(0)}(x), \xi^{(0)\prime}(x)) < 0$.

Now we are ready to prove our optimality result.

Theorem 3

For every $b > 2$, there exists a positive c such that $\kappa^*(J_c, \Phi) = b$, and the bounded normal scores estimator J_c satisfies the first order necessary conditions for being a minimum of $V(J, \Phi)$, subject to $\kappa^*(J, \Phi) \leqslant b$.

Proof. The existence of J_c (given $b > 2$) follows directly from the previous lemma. Since J is determined up to a multiplicative positive constant and $V(J, \Phi) = A(\psi)/B^2(\psi)$, we have to show that

$$\psi_c(x) = \begin{cases} x & \text{if } |x| \leqslant c \\ c \cdot sign(x), & otherwise \end{cases} \tag{4.13}$$

satisfies the first order necessary conditions for being a minimum of $A(\psi)$, subject to

$$B(\psi) = \int \psi'(x)\phi(x)dx = B(\psi_c) \tag{4.14}$$

and

$$2[1 + (h(x; \psi) - \psi'(x))/B(\psi_c)] \leqslant b. \tag{4.15}$$

We shall now apply Theorem 2. Define

$$\xi_1(x) = \psi(x)$$

$$\xi_2(x) = \int_{-\infty}^{-x} \xi_1'(s) s\, \Phi(s) ds,$$

$$\xi_3(x) = \int_{-\infty}^{x} \xi_1'(s) s\, \Phi(s) ds,$$

$$l=3,\ m'=2,\ m=4,\ p=1,\ a=-\infty\ ,\ b=\infty,$$
$$L(x,\xi,\xi')=\xi_1'^2\cdot\phi(x),$$
$$L_1(x,\xi,\xi')=\xi_1'\cdot\phi(x)-B(\psi_c),$$
$$\gamma_1(x,\xi,\xi')=\xi_2+\xi_3-\xi_1'-K_c\ ,$$
$$\gamma_2(x,\xi,\xi')=-\xi_1',$$
$$\gamma_3(x,\xi,\xi')=\xi_2'-\xi_1'\cdot x\cdot[1-\Phi(x)],$$
$$\gamma_4(x,\xi,\xi')=\xi_3'-\xi_1'\cdot x\cdot\Phi(x),$$

where $K_c=(\frac{1}{2}b-1)B(\psi_c)$. An easy calculation shows that

$$E(x,\xi,\xi',\mu,\mu)\equiv 0 \quad \text{for } all\ (x,\xi,\xi',\mu,\mu). \tag{4.16}$$

Moreover, from (4.8) we obtain

$$[2\lambda_0\xi_1^{(0)'}+(\lambda_1-\mu_3+\mu_4)x]\phi(x)=-\mu_1'-\mu_2'-\mu_1\cdot x-\mu_3-(-\mu_3+\mu_4)\Phi(x),\ \text{for } all\ x \tag{4.17}$$

$$\mu_3'=\mu_4'=\mu_1, \tag{4.18}$$

$$\gamma_1(x,\xi^{(0)'},\xi^{(0)''})\cdot\mu_1(x)=\xi_1^{(0)'}\cdot\mu_2(x). \tag{4.19}$$

Finally, from (4.9), using (4.16), we get

$$\mu_1(x)\cdot\gamma_1(x,\xi^{(0)'}(x),\xi^{(0)''}(x))-\mu_2(x)\cdot\xi_1^{(0)'}(x)\leqslant 0. \tag{4.20}$$

First consider the x's for which $\gamma_1(x,\xi^{(0)'},\xi^{(0)''})<0$.
In this region we choose

$$\mu_1(x)\equiv 0,\ \mu_2(x)\equiv c_2>0,\ \lambda_1=-2\lambda_0,\ \lambda_0>0.$$

Then from (4.18)

$$\mu_3(x)\equiv c_3,\quad \mu_4(x)\equiv c_4,$$

and from (4.17)

$$c_3=-c_4=0,$$
$$\xi_1^{(0)'}(x)=x.$$

Therefore, $\xi_1^{(0)'}(x)=\psi_c(x)$ for all x's for which the change-of-variance function is strictly less than b.

Secondly, consider the x's for which $\gamma_1(x,\xi^{(0)'},\xi^{(0)''})=0$.
In this region we choose

$$\mu_1(x)\equiv c_1>0,\quad \mu_2(x)\neq 0.$$

Then

$$\xi_1^{(0)''}(x)=0,\ i.e.\ \ \xi_1^{(0)'}(x)=c\ ,$$

and

$$\mu_3(x)=\mu_4(x)=c_1\cdot x$$
$$\mu_2(x)=-2\lambda_0\Phi(x)-\lambda_1\phi(x)-c_1\cdot x^2.$$

Therefore, $\xi_1^{(0)'}(x)=\psi_c(x)$ where the change-of-variance function equals the bound b.

This completes the proof.

Table 1 gives some numerical values for the bounded normal scores estimator. In particular, for a given c one can read the change-of-variance sensitivity, the efficiency at the normal model and an approximation (using (2.7)) of the maximum asymptotic variance over a neighborhood of the normal compared to the asymptotic variance at the normal model. From this table one can clearly see the trade-off between robustness and efficiency. (Note that $c=0$ corresponds to the median, and $c=\infty$ corresponds to the normal scores estimator.)

Table 1

c	$\kappa^*(J_c,\Phi)$	$\mathrm{eff}(J_c,\Phi)$	$\exp(\varepsilon\kappa^*(J_c,\Phi))$ ($\equiv \sup\{V(J_c,G):G\in P_\varepsilon(\Phi)\}/V(J_c,\Phi)$)			
			$\varepsilon=.01$	$\varepsilon=.025$	$\varepsilon=.05$	$\varepsilon=.1$
0.	2.000	.6366	1.020	1.051	1.105	1.221
0.2	2.027	.7017	1.020	1.051	1.107	1.224
0.4	2.108	.7636	1.021	1.054	1.111	1.235
0.6	2.246	.8184	1.023	1.058	1.119	1.252
0.8	2.444	.8649	1.025	1.063	1.130	1.277
1.0	2.709	.9031	1.027	1.070	1.145	1.311
1.2	3.045	.9332	1.030	1.079	1.164	1.356
1.4	3.460	.9555	1.035	1.090	1.189	1.413
1.6	3.959	.9716	1.040	1.104	1.‿19	1.486
1.8	4.546	.9825	1.046	1.120	1.255	1.576
2.0	5.226	.9898	1.054	1.140	1.299	1.686
∞	∞	1.				

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